Preface

Non-commutative probability theory is a branch of generalized probability theory that studies the probability of events in non-commutative algebras (e.g. the algebra of observables in quantum mechanics). In the 20th century, non-commutative probability theory has been applied to the study of quantum mechanics as the classical probability theory is not enough to describe quantum mechanics [KM].

Recently, the concentration of measure phenomenon has been applied to the study of non-commutative probability theory. Basically, the non-trivial observation, citing from Gromov's work [Gro81], states that an arbitrary 1-Lipschitz function $f : S^n \to \mathbb{R}$ concentrates near a single value $a_0 \in \mathbb{R}$ as strongly as the distance function does. That is,

$$\mu\{x \in S^n : |f(x) - a_0| \geq \epsilon\} < \kappa_n(\epsilon) \leq 2\exp\left(-\frac{(n-1)\epsilon^2}{2}\right)$$

is applied to computing the probability that, given a bipartite system $A \otimes B$, assume $\dim(B) \geq \dim(A) \geq 3$, as the dimension of the smaller system A increases, with very high probability, a random pure state $\sigma = |\psi\rangle\langle\psi|$ selected from $A \otimes B$ is almost as good as the maximally entangled state.

Mathematically, that is:

Let $\psi \in \mathcal{P}(A \otimes B)$ be a random pure state on $A \otimes B$.

If we define $\beta = \frac{1}{\ln(2)}\frac{d_A}{d_B}$, then we have

$$\Pr[H(\psi_A) < \log_2(d_A) - \alpha - \beta] \leq \exp\left(-\frac{1}{8\pi^2\ln(2)}\frac{(d_A d_B - 1)\alpha^2}{(\log_2(d_A))^2}\right)$$

where $d_B \geq d_A \geq 3$ [HLW06].

In this report, we will show the process of my exploration of the concentration of measure phenomenon in the context of non-commutative probability theory. We assume the reader is an undergraduate student in mathematics and is familiar with the basic concepts of probability theory, measure theory, linear algebra, and some basic skills of mathematical analysis. To make the report more self-contained, we will add detailed annotated proofs that I understand and references for the original sources.

How to use the dependency graph

Since our topic integrates almost everything I've learned during undergraduate study, I will try to make some dependency graph for reader and for me to keep track of what are the necessary knowledge to understand part of the report.

One can imagine the project as a big tree, where the root is in undergrad math and branches out to the topics of the report, including many advanced topics and motivation to study them.
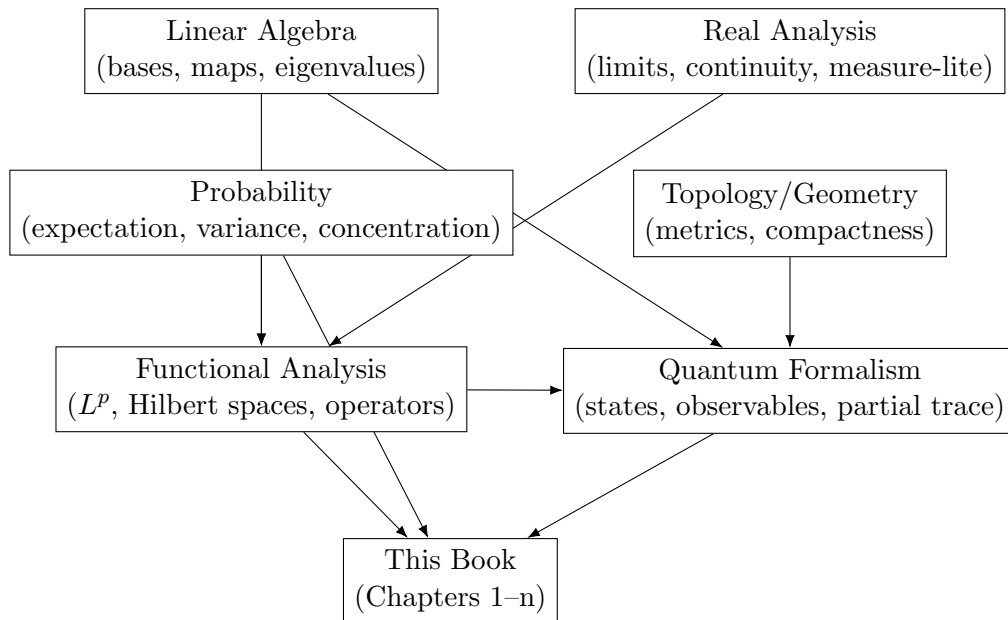
Figure 1: Dependency tree: prerequisites and how they feed into the main text.

References

[Gro81]   M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces.* Birkhäuser, 1981.

[HLW06]   P. Hayden, D. W. Leung, and A. Winter. "Aspects of Generic Entanglement". In: *Communications in Mathematical Physics* 265.1 (Mar. 2006), pp. 95–117. ISSN: 1432-0916. DOI: 10.1007/s00220-006-1535-6. URL: http://dx.doi.org/10.1007/s00220-006-1535-6.

[KM]   B. Kümmer and H. Maassen. "Elements of quantum probability". In: *Quantum Probability Communications*, pp. 73–100. DOI: 10.1142/9789812816054_0003. URL: https://www.worldscientific.com/doi/abs/10.1142/9789812816054_0003.