Concentration of Measure And Quantum Entanglement

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Chapter 1

Concentration of Measure And Quantum Entanglement

First, we will build the mathematical model describing the behavior of quantum system and why they makes sense for physicists and meaningful for general publics.

1.1 Motivation

First, we introduce a motivation for introducing non-commutative probability theory to the study of quantum mechanics. This section is mainly based on the book [KM].

1.1.1 Light polarization and the violation of Bell's inequality

The light which comes through a polarizer is polarized in a certain direction. If we fix the first filter and rotate the second filter, we will observe the intensity of the light will change.

The light intensity decreases with α (the angle between the two filters). The light should vanish when $\alpha = \pi/2$.

However, for a system of 3 polarizing filters F_1, F_2, F_3, having directions $\alpha_1, \alpha_2, \alpha_3$, if we put them on the optical bench in pairs, then we will have three random variables P_1, P_2, P_3.

Theorem 1. *Bell's 3 variable inequality:*

For any three random variables P_1, P_2, P_3 in a classical probability space, we have

$$\mathrm{Prob}(P_1 = 1, P_3 = 0) \leq \mathrm{Prob}(P_1 = 1, P_2 = 0) + \mathrm{Prob}(P_2 = 1, P_3 = 0)$$

Proof. By the law of total probability there are only two possibility if we don't observe any light passing the filter pair F_i, F_j, it means the photon is either blocked by F_i or F_j, it means

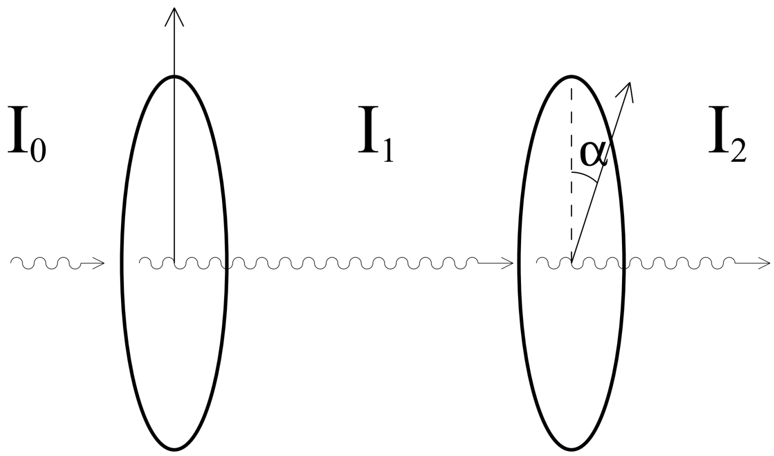

FIG. 1

Figure 1.1: The light polarization experiment, image from [KM]

$$\begin{aligned}
\text{Prob}(P_1 = 1, P_3 = 0) &= \text{Prob}(P_1 = 1, P_2 = 0, P_3 = 0) \\
&+ \text{Prob}(P_1 = 1, P_2 = 1, P_3 = 0) \\
&\leq \text{Prob}(P_1 = 1, P_2 = 0) + \text{Prob}(P_2 = 1, P_3 = 0)
\end{aligned}$$

□

However, according to our experimental measurement, for any pair of polarizers F_i, F_j, by the complement rule, we have

$$\begin{aligned}
\text{Prob}(P_i = 1, P_j = 0) &= \text{Prob}(P_i = 1) - \text{Prob}(P_i = 1, P_j = 1) \\
&= \frac{1}{2} - \frac{1}{2}\cos^2(\alpha_i - \alpha_j) \\
&= \frac{1}{2}\sin^2(\alpha_i - \alpha_j)
\end{aligned}$$

This leads to a contradiction if we apply the inequality to the experimental data.

$$\frac{1}{2}\sin^2(\alpha_1 - \alpha_3) \leq \frac{1}{2}\sin^2(\alpha_1 - \alpha_2) + \frac{1}{2}\sin^2(\alpha_2 - \alpha_3)$$

If $\alpha_1 = 0, \alpha_2 = \frac{\pi}{6}, \alpha_3 = \frac{\pi}{3}$, then

$$\begin{aligned}
\frac{1}{2}\sin^2(-\frac{\pi}{3}) &\leq \frac{1}{2}\sin^2(-\frac{\pi}{6}) + \frac{1}{2}\sin^2(\frac{\pi}{6} - \frac{\pi}{3}) \\
\frac{3}{8} &\leq \frac{1}{8} + \frac{1}{8} \\
\frac{3}{8} &\leq \frac{1}{4}
\end{aligned}$$

Other revised experiments (e.g., Aspect's experiment, calcium entangled photon experiment) are also conducted and the inequality is still violated.

1.1.2 The true model of light polarization

The full description of the light polarization is given below:

State of polarization of a photon: $\psi = \alpha|0\rangle + \beta|1\rangle$, where $|0\rangle$ and $|1\rangle$ are the two orthogonal polarization states in $\mathbb{C}^2$.

Polarization filter (generalized 0,1 valued random variable): orthogonal projection P_α on $\mathbb{C}^2$ corresponding to the direction α (operator satisfies $P_\alpha^* = P_\alpha = P_\alpha^2$).

The matrix representation of P_α is given by

$$P_\alpha = \begin{pmatrix} \cos^2(\alpha) & \cos(\alpha)\sin(\alpha) \\ \cos(\alpha)\sin(\alpha) & \sin^2(\alpha) \end{pmatrix}$$

Probability of a photon passing through the filter P_α is given by $\langle P_\alpha \psi, \psi \rangle$; this is $\cos^2(\alpha)$ if we set $\psi = |0\rangle$.

Since the probability of a photon passing through the three filters is not commutative, it is impossible to discuss $\text{Prob}(P_1 = 1, P_3 = 0)$ in the classical setting.

1.2 Concentration of measure phenomenon

Definition 2. *η-Lipschitz function*

Let (X, dist_X) and (Y, dist_Y) be two metric spaces. A function $f : X \to Y$ is said to be η-Lipschitz if there exists a constant $L \in \mathbb{R}$ such that

$$\text{dist}_Y(f(x), f(y)) \leq L\, \text{dist}_X(x, y)$$

for all $x, y \in X$. And $\eta = \|f\|_{\text{Lip}} = \inf_{L \in \mathbb{R}} L$.

That basically means that the function f should not change the distance between any two pairs of points in X by more than a factor of L.

This is a stronger condition than continuity, every Lipschitz function is continuous, but not every continuous function is Lipschitz.

Lemma 3. *Isoperimetric inequality on the sphere:*

Let $\sigma_n(A)$ denote the normalized area of A on the n-dimensional sphere S^n. That is, $\sigma_n(A) := \frac{\text{Area}(A)}{\text{Area}(S^n)}$.

Let $\epsilon > 0$. Then for any subset $A \subset S^n$, given the area $\sigma_n(A)$, the spherical caps minimize the volume of the ϵ-neighborhood of A.

Suppose $\sigma^n(\cdot)$ is the normalized volume measure on the sphere $S^n(1)$, then for any closed subset $\Omega \subset S^n(1)$, we take a metric ball B_Ω of $S^n(1)$ with $\sigma^n(B_\Omega) = \sigma^n(\Omega)$. Then we have

$$\sigma^n(U_r(\Omega)) \geq \sigma^n(U_r(B_\Omega))$$

where $U_r(A) = \{x \in X : d(x, A) < r\}$

Intuitively, the lemma means that the spherical caps are the most efficient way to cover the sphere.

Here, the efficiency is measured by the epsilon-neighborhood of the boundary of the spherical cap.

To prove the lemma, we need to have a good understanding of the Riemannian geometry of the sphere. For now, let's just take the lemma for granted.

1.2.1 Levy's concentration theorem

Theorem 4. *Levy's concentration theorem:*

An arbitrary 1-Lipschitz function $f : S^n \to \mathbb{R}$ *concentrates near a single value* $a_0 \in \mathbb{R}$ *as strongly as the distance function does.*

That is,

$$\mu\{x \in S^n : |f(x) - a_0| \geq \epsilon\} < \kappa_n(\epsilon) \leq 2\exp\left(-\frac{(n-1)\epsilon^2}{2}\right)$$

where

$$\kappa_n(\epsilon) = \frac{\int_\epsilon^{\frac{\pi}{2}} \cos^{n-1}(t)dt}{\int_0^{\frac{\pi}{2}} \cos^{n-1}(t)dt}$$

a_0 *is the* ***Levy mean*** *of function* f*, that is, the level set* $f^{-1} : \mathbb{R} \to S^n$ *divides the sphere into equal halves, characterized by the following equality:*

$$\mu(f^{-1}(-\infty, a_0]) \geq \frac{1}{2} \text{ and } \mu(f^{-1}[a_0, \infty)) \geq \frac{1}{2}$$

We will prove the theorem via the Maxwell-Boltzmann distribution law. [Shi14]

Definition 5. *Gaussian measure:*

We denote the Gaussian measure on $\mathbb{R}^k$ *as* γ^k*.*

$$d\gamma^k(x) := \frac{1}{\sqrt{2\pi}^k} \exp(-\frac{1}{2}\|x\|^2)dx$$

$x \in \mathbb{R}^k$, $\|x\|^2 = \sum_{i=1}^k x_i^2$ *is the Euclidean norm, and* dx *is the Lebesgue measure on* $\mathbb{R}^k$*.*

Basically, you can consider the Gaussian measure as the normalized Lebesgue measure on $\mathbb{R}^k$ with standard deviation 1.

It also has another name, the Projective limit theorem. [Ver18]

If $X \sim \text{Unif}(S^n(\sqrt{n}))$, then for any fixed unit vector x we have $\langle X, x\rangle \to N(0,1)$ in distribution as $n \to \infty$.

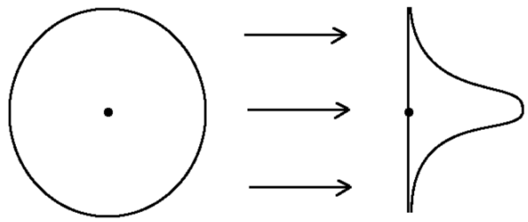

Figure 3.9 The projective central limit theorem: the projection of the uniform distribution on the sphere of radius $\sqrt{n}$ onto a line converges to the normal distribution $N(0,1)$ as $n \to \infty$.

Figure 1.2: Maxwell-Boltzmann distribution law, image from [Ver18]

Lemma 6. *Maxwell-Boltzmann distribution law:*

For any natural number k,

$$\frac{d(\pi_{n,k})_* \sigma^n(x)}{dx} \to \frac{d\gamma^k(x)}{dx}$$

where $(\pi_{n,k})_* \sigma^n$ is the push-forward measure of σ^n by $\pi_{n,k}$.

In other words,

$$(\pi_{n,k})_* \sigma^n \to \gamma^k \text{ weakly as } n \to \infty$$

Proof. We denote the n-dimensional volume measure on $\mathbb{R}^k$ as vol_k.

Observe that $\pi_{n,k}^{-1}(x), x \in \mathbb{R}^k$ is isometric to $S^{n-k}(\sqrt{n - \|x\|^2})$, that is, for any $x \in \mathbb{R}^k$, $\pi_{n,k}^{-1}(x)$ is a sphere with radius $\sqrt{n - \|x\|^2}$ (by the definition of $\pi_{n,k}$).

So,

$$\begin{aligned} \frac{d(\pi_{n,k})_* \sigma^n(x)}{dx} &= \frac{\mathrm{vol}_{n-k}(\pi_{n,k}^{-1}(x))}{\mathrm{vol}_k(S^n(\sqrt{n}))} \\ &= \frac{(n - \|x\|^2)^{\frac{n-k}{2}}}{\int_{\|x\| \le \sqrt{n}} (n - \|x\|^2)^{\frac{n-k}{2}} dx} \end{aligned}$$

as $n \to \infty$.

Note that $\lim_{n\to\infty}(1 - \frac{a}{n})^n = e^{-a}$ for any $a > 0$.

$(n - \|x\|^2)^{\frac{n-k}{2}} = \left(n(1 - \frac{\|x\|^2}{n})\right)^{\frac{n-k}{2}} \to n^{\frac{n-k}{2}} \exp(-\frac{\|x\|^2}{2})$

So

$$\frac{(n-\|x\|^2)^{\frac{n-k}{2}}}{\int_{\|x\|\le\sqrt{n}}(n-\|x\|^2)^{\frac{n-k}{2}}\,dx} = \frac{e^{-\frac{\|x\|^2}{2}}}{\int_{x\in\mathbb{R}^k} e^{-\frac{\|x\|^2}{2}}\,dx} = \frac{1}{(2\pi)^{\frac{k}{2}}}e^{-\frac{\|x\|^2}{2}} = \frac{d\gamma^k(x)}{dx}$$

□

Now we can prove Levy's concentration theorem, the proof is from [Shi14].

Proof. Let $f_n : S^n(\sqrt{n}) \to \mathbb{R}$, $n = 1, 2, \ldots$, be 1-Lipschitz functions.

Let x and x' be two given real numbers and $\gamma^1(-\infty, x] = \overline{\sigma}_\infty[-\infty, x']$, suppose $\sigma_\infty\{x'\} = 0$, where $\{\sigma_i\}$ is a sequence of Borel probability measures on $\mathbb{R}$.

We want to show that, for all non-negative real numbers ϵ_1 and ϵ_2.

$$\sigma_\infty[x'-\epsilon_1, x'+\epsilon_2] \ge \gamma^1[x-\epsilon_1, x+\epsilon_2]$$

Consider the two spherical cap $\Omega_+ := \{f_{n_i} \ge x'\}$ and $\Omega_- := \{f_{n_i} \le x\}$. Note that $\Omega_+ \cup \Omega_- = S^{n_i}(\sqrt{n_i})$.

It is sufficient to show that,

$$U_{\epsilon_1}(\Omega_+) \cup U_{\epsilon_2}(\Omega_-) \subset \{x'-\epsilon_1 \le f_{n_i} \le x'+\epsilon_2\}$$

By 1-Lipschitz continuity of f_{n_i}, we have for all $\zeta \in U_{\epsilon_1}(\Omega_+)$, there is a point $\xi \in \Omega_+$ such that $d(\zeta, \xi) \le \epsilon_1$. So $U_{\epsilon_1}(\Omega_+) \subset \{f_{n_i} \ge x'-\epsilon_1\}$. With the same argument, we have $U_{\epsilon_2}(\Omega_-) \subset \{f_{n_i} \le x+\epsilon_2\}$.

So the push-forward measure of $(f_{n_i})_*\sigma^{n_i}$ of $[x'-\epsilon_1, x'+\epsilon_2]$ is

$$\begin{aligned}(f_{n_i})_*\sigma^{n_i}[x'-\epsilon_1, x'+\epsilon_2] &= \sigma^{n_i}(x'-\epsilon_1 \le f_{n_i} \le x'+\epsilon_2) \\ &\ge \sigma^{n_i}(U_{\epsilon_1}(\Omega_+) \cap U_{\epsilon_2}(\Omega_-)) \\ &= \sigma^{n_i}(U_{\epsilon_1}(\Omega_+)) + \sigma^{n_i}(U_{\epsilon_2}(\Omega_-)) - 1\end{aligned}$$

By the lemma 3, we have

$$\sigma^{n_i}(U_{\epsilon_1}(\Omega_+)) \ge \sigma^{n_i}(U_{\epsilon_1}(B_{\Omega_+})) \quad \text{and} \quad \sigma^{n_i}(U_{\epsilon_2}(\Omega_-)) \ge \sigma^{n_i}(U_{\epsilon_2}(B_{\Omega_-}))$$

By the lemma 6, we have

$$\sigma^{n_i}(U_{\epsilon_1}(\Omega_+)) + \sigma^{n_i}(U_{\epsilon_2}(\Omega_-)) \to \gamma^1[x'-\epsilon_1, x'+\epsilon_2] + \gamma^1[x-\epsilon_1, x+\epsilon_2]$$

Therefore,

$$\begin{aligned}\sigma_\infty[x' - \epsilon_1, x' + \epsilon_2] &\geq \liminf_{i\to\infty} (f_{n_i})_* \sigma^{n_i}[x' - \epsilon_1, x' + \epsilon_2] \\ &\geq \gamma^1[x' - \epsilon_1, \infty) \cap \gamma^1(-\infty, x + \epsilon_2] - 1 \\ &= \gamma^1[x - \epsilon_1, x + \epsilon_2]\end{aligned}$$

□

The full proof of Levy's concentration theorem requires more digestion for cases where $\overline{\sigma}_\infty \neq \delta_{\pm\infty}$ but I don't have enough time to do so. This section may be filled in the next semester.

1.3 The application of the concentration of measure phenomenon in non-commutative probability theory

In quantum communication, we can pass classical bits by sending quantum states. However, by the indistinguishability (Proposition **??**) of quantum states, we cannot send an infinite number of classical bits over a single qubit. There exists a bound for zero-error classical communication rate over a quantum channel.

Theorem 7. *Holevo bound:*

The maximal amount of classical information that can be transmitted by a quantum system is given by the Holevo bound. $\log_2(d)$ *is the maximum amount of classical information that can be transmitted by a quantum system with d levels (that is, basically, the number of qubits).*

The proof of the Holevo bound can be found in [NC10]. In current state of the project, this theorem is not heavily used so we will not make annotated proof here.

1.3.1 Quantum communication

To surpass the Holevo bound, we need to use the entanglement of quantum states.

Definition 8. *Bell state:*

The Bell states are the following four states:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

These are a basis of the 2-qubit Hilbert space.

1.3.2 Superdense coding and entanglement

The description of the superdense coding can be found in [GMS15] and [Hay10].

Suppose A and B share a Bell state (or other maximally entangled state) $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, where A holds the first part and B holds the second part.

A wishes to send 2 **classical bits** to B.

A performs one of four Pauli unitaries (some fancy quantum gates named X, Y, Z, I) on the combined state of entangled qubits $\otimes$ one qubit. Then A sends the resulting one qubit to B.

This operation extends the initial one entangled qubit to a system of one of four orthogonal Bell states.

B performs a measurement on the combined state of the one qubit and the entangled qubits he holds.

B decodes the result and obtains the 2 classical bits sent by A.

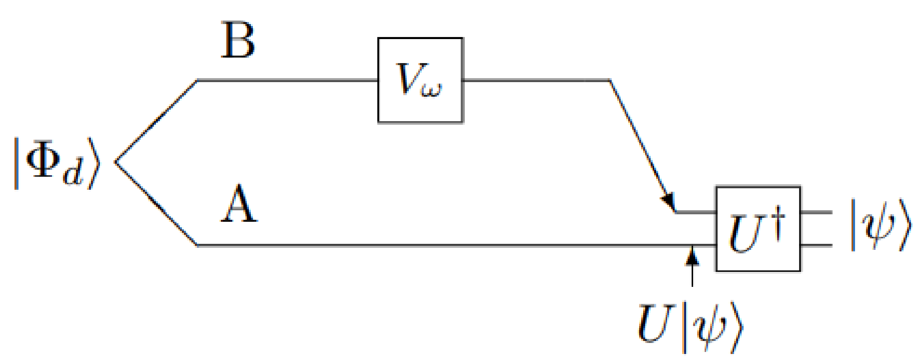

Figure 1.3: Superdense coding, image from [Hay10]

Note that superdense coding is a way to send 2 classical bits of information by sending 1 qubit with 1 entangled qubit. **The role of the entangled qubit** is to help them to distinguish the 4 possible states of the total 3 qubits system where 2 of them (the pair of entangled qubits) are mathematically the same.

Additionally, no information can be gained by measuring a pair of entangled qubits. To send information from A to B, we need to physically send the qubits from A to B. That means, we cannot send information faster than the speed of light.

1.3.3 Hayden's concentration of measure phenomenon

The application of the concentration of measure phenomenon in the superdense coding can be realized in random sampling the entangled qubits [Hay10]:

It is a theorem connecting the following mathematical structure:

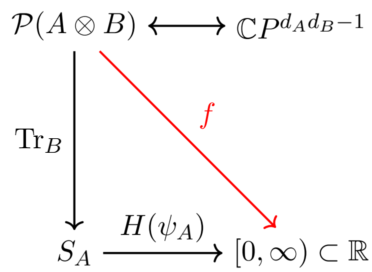

Figure 1.4: Mathematical structure for Hayden's concentration of measure phenomenon

- The red arrow is the concentration of measure effect. $f = H(\mathrm{Tr}_B(\psi))$.
- S_A denotes the mixed states on A.

To prove the concentration of measure phenomenon, we need to analyze the following elements involved in figure 1.4:

First, we need to define what is a random state in a bipartite system. In fact, for pure states, there is a unique uniform distribution under Haar measure that is unitarily invariant.

$U(n)$ is the group of all $n \times n$ **unitary matrices** over $\mathbb{C}$,

$$U(n) = \{A \in \mathbb{C}^{n \times n} : A^* A = AA^* = I_n\}$$

The uniqueness of such measurement came from the lemma below [Mec]

Lemma 9. *Let $(U(n), \|\cdot\|, \mu)$ be a metric measure space where $\|\cdot\|$ is the Hilbert-Schmidt norm and μ is the measure function.*

The Haar measure on $U(n)$ is the unique probability measure that is invariant under the action of $U(n)$ on itself.

That is, fixing $B \in U(n)$, $\forall A \in U(n)$, $\mu(A \cdot B) = \mu(B \cdot A) = \mu(B)$.

The Haar measure is the unique probability measure that is invariant under the action of $U(n)$ on itself.

The existence and uniqueness of the Haar measure is a theorem in compact lie group theory. For this research topic, we will not prove it.

A random pure state φ is any random variable distributed according to the unitarily invariant probability measure on the pure states $\mathcal{P}(A)$ of the system A, denoted by $\varphi \in_R \mathcal{P}(A)$.

It is trivial that for the space of pure state, we can easily apply the Haar measure as the unitarily invariant probability measure since the space of pure state is S^n for some n. However, for the case of mixed states, that is a bit complicated and we need to use partial tracing to defined the rank-s random states.

Definition 10. *Rank-s random state.*

For a system A and an integer $s \geq 1$, consider the distribution onn the mixed states $\mathcal{S}(A)$ of A induced by the partial trace over the second factor form the uniform distribution on pure states of $A \otimes \mathbb{C}^s$. Any random variable ρ distributed as such will be called a rank-s random states; denoted as $\rho \in_R \mathcal{S}_s(A)$. And $\mathcal{P}(A) = \mathcal{S}_1(A)$.

Due to time constrains of the projects, the following lemma is demonstrated but not investigated thoroughly through the research:

Lemma 11. *Page's lemma for expected entropy of mixed states*

Choose a random pure state $\sigma = |\psi\rangle\langle\psi|$ from $A' \otimes A$.

The expected value of the entropy of entanglement is known and satisfies a concentration inequality known as Page's formula [Pag; San95; BŻ17][15.72]. The detailed proof is not fully explored in this project and is intended to be done in the next semester.

$$\mathbb{E}[H(\psi_A)] \geq \log_2(d_A) - \frac{1}{2\ln(2)}\frac{d_A}{d_B}$$

It basically provides a lower bound for the expected entropy of entanglement. Experimentally, we can have the following result (see Figure 1.5):

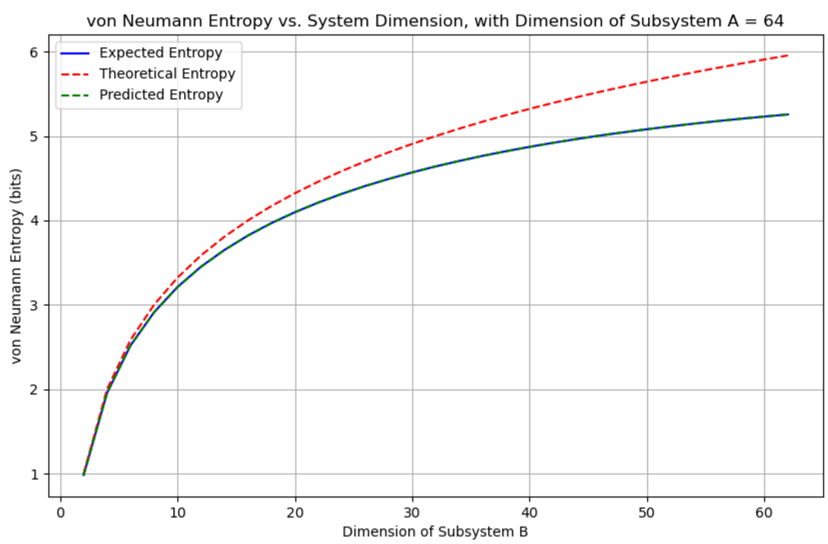

Figure 1.5: Entropy vs dimension

Then we have bound for Lipschitz constant η of the map $H(\varphi_A)$

Lemma 12. *The Lipschitz constant η of $S(\varphi_A)$ is upper bounded by $\sqrt{8}\log_2(d_A)$ for $d_A \geq 3$.*

From Levy's lemma, we have

If we define $\beta = \frac{1}{\ln(2)} \frac{d_A}{d_B}$, then we have

$$\Pr[H(\psi_A) < \log_2(d_A) - \alpha - \beta] \leq \exp\left(-\frac{1}{8\pi^2 \ln(2)} \frac{(d_A d_B - 1)\alpha^2}{(\log_2(d_A))^2} \right)$$

where $d_B \geq d_A \geq 3$ [HLW06].

Experimentally, we can have the following result:

As the dimension of the Hilbert space increases, the chance of getting an almost maximally entangled state increases (see Figure 1.6).

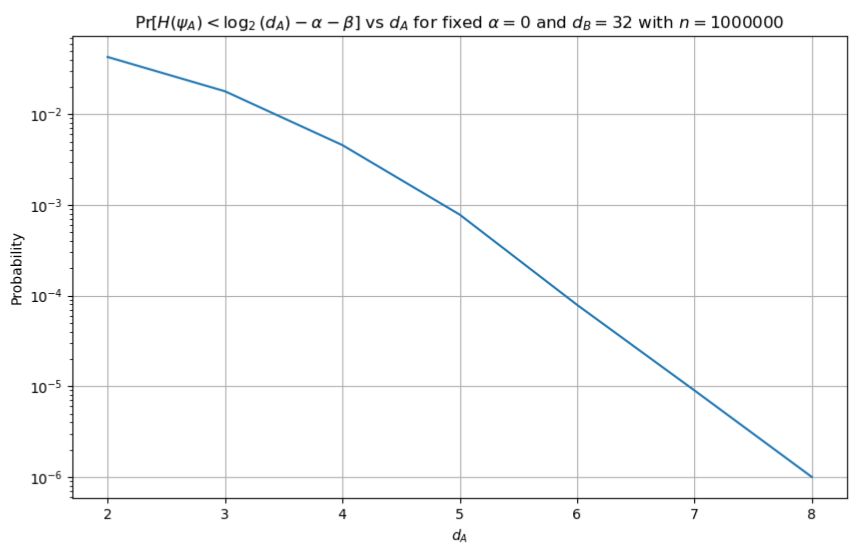

Figure 1.6: Entropy vs d_A

Chapter 2

Levy's family and observable diameters

In this section, we will explore how the results from Hayden's concentration of measure theorem can be understood in terms of observable diameters from Gromov's perspective and what properties it reveals for entropy functions.

2.1 Observable diameters

Chapter 3

Seigel-Bargmann Space

In this chapter, we will collect ideas and other perspective we have understanding the concentration of measure phenomenon. Especially with symmetric product of $\mathbb{C}P^1$ and see how it relates to Riemman surfaces and Seigel-Bargmann spaces.

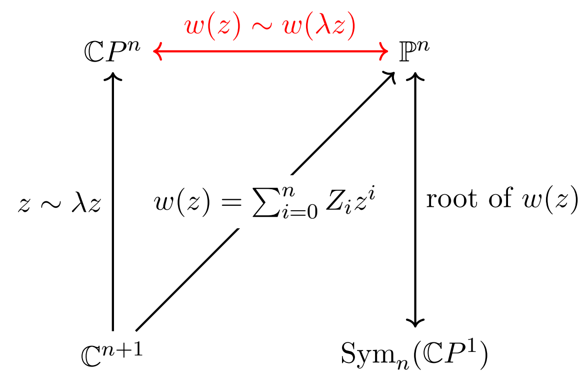

Figure 3.1: Majorana stellar representation

Basically, there is a bijection between the complex projective space $\mathbb{C}P^n$ and the set of roots of a polynomial of degree n.

We can use a symmetric group of permutations of n complex numbers (or S^2) to represent the $\mathbb{C}P^n$, that is, $\mathbb{C}P^n = S^2 \times S^2 \times \cdots \times S^2/S_n$.

One might be interested in the random sampling over the $\mathrm{Sym}_n(\mathbb{C}P^1)$ and the concentration of measure phenomenon on that.

3.1 Majorana stellar representation of the quantum state

3.2 Space of complex valued functions and pure states

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