Chapter 1

Levy's family and observable diameters

In this section, we will explore how the results from Hayden's concentration of measure theorem can be understood in terms of observable diameters from Gromov's perspective and what properties it reveals for entropy functions.

We will try to use the results from previous sections to estimate the observable diameter for complex projective spaces.

1.1 Observable diameters

Recall from previous sections, an arbitrary 1-Lipschitz function $f : S^n \to \mathbb{R}$ concentrates near a single value $a_0 \in \mathbb{R}$ as strongly as the distance function does.

Definition 1. *Let X be a topological space with the following:*

1. *X is a complete (every Cauchy sequence converges)*
2. *X is a metric space with metric d_X*
3. *X has a Borel probability measure μ_X*

Then (X, d_X, μ_X) is a ***metric measure space****.*

Definition 2. *Let (X, d_X) be a metric space. The* ***diameter*** *of a set $A \subset X$ is defined as*

$$\operatorname{diam}(A) = \sup_{x,y \in A} d_X(x, y).$$

Definition 3. *Let (X, d_X, μ_X) be a metric measure space, For any real number $\alpha \leq 1$, the* ***partial diameter*** *of X is defined as*

$$\operatorname{diam}(A; \alpha) = \inf_{A \subseteq X | \mu_X(A) \geq \alpha} \operatorname{diam}(A).$$

This definition generalize the relation between the measure and metric in the metric-measure space. Intuitively, the space with smaller partial diameter can take more volume with the same diameter constrains.

However, in higher dimensions, the volume may tend to concentrates more around a small neighborhood of the set, as we see in previous chapters with high dimensional sphere as example. We can safely cut $\kappa > 0$ volume to significantly reduce the diameter, this yields better measure for concentration for shapes in spaces with high dimension.

Definition 4. *Let X be a metric-measure space, Y be a metric space, and $f : X \to Y$ be a 1-Lipschitz function. Then $f_*\mu_X = \mu_Y$ is a push forward measure on Y.*

*For any real number $\kappa > 0$, the **κ-observable diameter with screen** Y is defined as*

$$\mathrm{ObserDiam}_Y(X;\kappa) = \sup\{\mathrm{diam}(f_*\mu_X; 1-\kappa)\}$$

*And the **obbservable diameter with screen** Y is defined as*

$$\mathrm{ObserDiam}_Y(X) = \inf_{\kappa>0} \max\{\mathrm{ObserDiam}_Y(X;\kappa)\}$$

*If $Y = \mathbb{R}$, we call it the **observable diameter**.*

If we collapse it naively via

$$\inf_{\kappa>0} \mathrm{ObserDiam}_Y(X;\kappa),$$

we typically get something degenerate: as $\kappa \to 1$, the condition "mass $\geq 1-\kappa$" becomes almost empty space, so $\mathrm{diam}(\nu; 1-\kappa)$ can be forced to be 0 (take a tiny set of positive mass), and hence the infimum tends to 0 for essentially any non-atomic space.

This is why one either:

1. keeps $\mathrm{ObserDiam}_Y(X;\kappa)$ as a *function of* κ (picking κ to be small but not 0), or
2. if one insists on a single number, balances "spread" against "exceptional mass" by defining $\mathrm{ObserDiam}_Y(X) = \inf_{\kappa>0} \max\{\mathrm{ObserDiam}_Y(X;\kappa), \kappa\}$ as above.

The point of the $\max\{\cdot,\kappa\}$ is that it prevents cheating by taking κ close to 1: if κ is large then the maximum is large regardless of how small $\mathrm{ObserDiam}_Y(X;\kappa)$ is, so the infimum is forced to occur where the exceptional mass and the observable spread are small.

Few additional proposition in [Shi14] will help us to estimate the observable diameter for complex projective spaces.

Proposition 5. *Let X and Y be two metric-measure spaces and $\kappa > 0$, and let $f : Y \to X$ be a 1-Lipschitz function (Y dominates X, denoted as $X \prec Y$) then:*

1. $\mathrm{diam}(X, 1-\kappa) \leq \mathrm{diam}(Y, 1-\kappa)$
2. $\mathrm{ObserDiam}(X; -\kappa) \leq \mathrm{diam}(X; 1-\kappa)$*, and* $\mathrm{ObserDiam}(X)$ *is finite.*
3. $\mathrm{ObserDiam}(X; -\kappa) \leq \mathrm{ObserDiam}(Y; -\kappa)$

Proof. Since f is 1-Lipschitz, we have $f_*\mu Y = \mu_X$. Let A be any Borel set of Y with $\mu_Y(A) \geq 1-\kappa$ and $\overline{f(A)}$ be the closure of $f(A)$ in X. We have $\mu_X(\overline{f(A)}) = \mu_Y(f^{-1}(\overline{f(A)})) \geq \mu_Y(A) \geq 1-\kappa$ and by the 1-lipschitz property, $\mathrm{diam}(\overline{f(A)}) \leq \mathrm{diam}(A)$, so $\mathrm{diam}(X; 1-\kappa) \leq \mathrm{diam}(A) \leq \mathrm{diam}(Y; 1-\kappa)$.

Let $g : X \to \mathbb{R}$ be any 1-lipschitz function, since $(\mathbb{R}, |\cdot|, g_*\mu_X)$ is dominated by X, $\mathrm{diam}(\mathbb{R}; 1-\kappa) \leq \mathrm{diam}(X; 1-\kappa)$. Therefore, $\mathrm{ObserDiam}(X; -\kappa) \leq \mathrm{diam}(X; 1-\kappa)$.

and

$$\mathrm{diam}(g_*\mu_X; 1-\kappa) \leq \mathrm{diam}((f \circ g)_*\mu_Y; 1-\kappa) \leq \mathrm{ObserDiam}(Y; 1-\kappa)$$

□

Proposition 6. *Let X be an metric-measure space. Then for any real number $t > 0$, we have*

$$\mathrm{ObserDiam}(tX; -\kappa) = t\,\mathrm{ObserDiam}(X; -\kappa)$$

Where $tX = (X, tdX, \mu X)$.

Proof.

$$\begin{aligned}
\mathrm{ObserDiam}(tX; -\kappa) &= \sup\{\mathrm{diam}(f_*\mu_X; 1-\kappa) | f : tX \to \mathbb{R} \text{ is 1-Lipschitz}\} \\
&= \sup\{\mathrm{diam}(f_*\mu_X; 1-\kappa) | t^{-1}f : X \to \mathbb{R} \text{ is 1-Lipschitz}\} \\
&= \sup\{\mathrm{diam}((tg)_*\mu_X; 1-\kappa) | g : X \to \mathbb{R} \text{ is 1-Lipschitz}\} \\
&= t\sup\{\mathrm{diam}(g_*\mu_X; 1-\kappa) | g : X \to \mathbb{R} \text{ is 1-Lipschitz}\} \\
&= t\,\mathrm{ObserDiam}(X; -\kappa)
\end{aligned}$$

□

1.1.1 Observable diameter for class of spheres

In this section, we will try to use the results from previous sections to estimate the observable diameter for class of spheres.

Theorem 7. *For any real number κ with $0 < \kappa < 1$, we have*

$$\mathrm{ObserDiam}(S^n(1); -\kappa) = O(\sqrt{n})$$

Proof. First, recall that by maxwell boltzmann distribution, we have that for any $n > 0$, let $I(r)$ denote the measure of standard gaussian measure on the interval $[0, r]$. Then we have

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{ObserDiam}(S^n(\sqrt{n}); -\kappa) &= \lim_{n\to\infty} \sup\{\mathrm{diam}((\pi_{n,k})_*\sigma^n; 1-\kappa) | \pi_{n,k} \text{ is 1-Lipschitz}\} \\
&= \lim_{n\to\infty} \sup\{\mathrm{diam}(\gamma^1; 1-\kappa) | \gamma^1 \text{ is the standard gaussian measure}\} \\
&= \mathrm{diam}(\gamma^1; 1-\kappa) \\
&= 2I^{-1}(\frac{1-\kappa}{2}) \text{ cutting the extremum for normal distribution}
\end{aligned}$$

By proposition 6, we have

$$\mathrm{ObserDiam}(S^n(\sqrt{n}); -\kappa) = \sqrt{n}\,\mathrm{ObserDiam}(S^n(1); -\kappa)$$

So $\mathrm{ObserDiam}(S^n(1); -\kappa) = \sqrt{n}(2I^{-1}(\frac{1-\kappa}{2})) = O(\sqrt{n})$. □

From the previous discussion, we see that the only remaining for finding observable diameter of $\mathbb{C}P^n$ is to find the lipchitz function that is isometric with consistent push-forward measure.

To find such metric, we need some additional results.

Definition 8. *Let M be a smooth manifold. A* ***Riemannian metric*** *on M is a smooth covariant tensor field $g \in \mathcal{T}^2(M)$ such that for each $p \in M$, g_p is an inner product on T_pM.*

$g_p(v, v) \geq 0$ for each $p \in M$ and each $v \in T_pM$. equality holds if and only if $v = 0$.

Theorem 9. *Let $(\tilde{M}, \tilde{g})$ be a Riemannian manifold, let $\pi : \tilde{M} \to M$ be a surjective smooth submersion, and let G be a group acting on $\tilde{M}$. If the* ***action*** *is*

1. *isometric: the map $x \mapsto \varphi \cdot x$ is an isometry for each $\varphi \in G$.*
2. *vertical: every element $\varphi \in G$ takes each fiber to itself, that is $\pi(\varphi \cdot p) = \pi(p)$ for all $p \in \tilde{M}$.*
3. *transitive on fibers: for each $p, q \in \tilde{M}$ such that $\pi(p) = \pi(q)$, there exists $\varphi \in G$ such that $\varphi \cdot p = q$.*

Then there is a unique Riemannian metric on M such that π is a Riemannian submersion.

A natural measure for $\mathbb{C}P^n$ is the normalized volume measure on $\mathbb{C}P^n$ induced from the Fubini-Study metric. [Lee18] Example 2.30

Definition 10. *Let n be a positive integer, and consider the complex projective space $\mathbb{C}P^n$ defined as the quotient space of $\mathbb{C}^{n+1}$ by the equivalence relation $z \sim z'$ if there exists $\lambda \in \mathbb{C}$ such that $z = \lambda z'$. The map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$ sending each point in $\mathbb{C}^{n+1} \setminus \{0\}$ to its span is surjective smooth submersion.*

Identifying $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$ with its Euclidean metric, we can view the unit sphere S^{2n+1} with its roud metric $\mathring{g}$ as an embedded Riemannian submanifold of $\mathbb{C}^{n+1} \setminus \{0\}$. Let $p : S^{2n+1} \to \mathbb{C}P^n$ denote the restriction of the map π. Then p is smooth, and its is surjective, because every 1-dimensional complex subspace contains elements of unit norm.

There are many additional properties for such construction, we will check them just for curiosity.

We need to show that it is a submersion.

Proof. Let $z_0 \in S^{2n+1}$ and set $\zeta_0 = p(z_0) \in \mathbb{C}P^n$. Since π is a smooth submersion, it has a smooth local section $\sigma : U \to \mathbb{C}^{n+1}$ defined on a neighborhood U of ζ_0 and satisfying $\sigma(\zeta_0) = z_0$ by the local section theorem (Theorem **??**). Let $v : \mathbb{C}^{n+1} \setminus \{0\} \to S^{2n+1}$ be the radial projection on to the sphere:

$$v(z) = \frac{z}{|z|}$$

Since dividing an element of $\mathbb{C}^{n+1}$ by a nonzero scalar does not change its span, it follows that $p \circ v = \pi$. Therefore, if we set $\tilde{\sigma} = v \circ \sigma$, then $\tilde{\sigma}$ is a smooth local section of p. Apply the local section theorem (Theorem **??**) again, this shows that p is a submersion.

Define an action of S^1 on S^{2n+1} by complex multiplication:

$$\lambda(z^1, z^2, \ldots, z^{n+1}) = (\lambda z^1, \lambda z^2, \ldots, \lambda z^{n+1})$$

for $\lambda \in S^1$ (viewed as complex number of norm 1) and $z = (z^1, z^2, \dots, z^{n+1}) \in S^{2n+1}$. This is easily seen to be isometric, vertical, and transitive on fibers of p.

By (Theorem 9). Therefore, there is a unique metric on $\mathbb{C}P^n$ such that the map $p : S^{2n+1} \to \mathbb{C}P^n$ is a Riemannian submersion. This metric is called the Fubini-study metric. □

1.1.2 Observable diameter for complex projective spaces

Using the projection map and Hopf's fibration, we can estimate the observable diameter for complex projective spaces from the observable diameter of spheres.

Theorem 11. *For any real number κ with $0 < \kappa < 1$, we have*

$$\mathrm{ObserDiam}(\mathbb{C}P^n(1); -\kappa) \leq O(\sqrt{n})$$

Proof. Recall from Example 2.30 in [Lee18], the Hopf fibration $f_n : S^{2n+1}(1) \to \mathbb{C}P^n$ is 1-Lipschitz continuous with respect to the Fubini-Study metric on $\mathbb{C}P^n$. and the push-forward $(f_n)_* \sigma^{2n+1}$ coincides with the normalized volume measure on $\mathbb{C}P^n$ induced from the Fubini-Study metric.

By proposition 5, we have $\mathrm{ObserDiam}(\mathbb{C}P^n(1); -\kappa) \leq \mathrm{ObserDiam}(S^{2n+1}(1); -\kappa) \leq O(\sqrt{n})$.

□

1.2 Use entropy function as estimator of observable diameter for complex projective spaces

In this section we describe a Monte Carlo pipeline for comparing concentration phenomena across three metric-measure spaces using real-valued entropy observables. The goal is not to compute the exact observable diameter

$$\mathrm{ObsDiam}_{\mathbb{R}}(X; -\kappa) = \sup_{f \in \mathrm{Lip}_1(X, \mathbb{R})} \mathrm{diam}(f_* \mu_X; 1 - \kappa),$$

but to estimate it by choosing a specific observable $f : X \to \mathbb{R}$ and then measuring the partial diameter of its push-forward distribution. Thus all numerical quantities below should be interpreted as *entropy-based observable-diameter proxies*, not exact observable diameters in Gromov's sense [Gro81; Shi14].

The screen is $\mathbb{R}$ equipped with the Euclidean metric, and for a fixed $\kappa \in (0, 1)$ we set

$$\alpha = 1 - \kappa.$$

Given sampled values $y_1, \dots, y_N \in \mathbb{R}$ of the observable, the code sorts them and computes the shortest interval $[a, b]$ containing at least $\lceil \alpha N \rceil$ samples. Its width

$$b - a$$

is the empirical partial diameter of the push-forward measure on $\mathbb{R}$.

To compare this width with the true observable diameter, the code also estimates an empirical Lipschitz constant of the chosen observable. If $x_i, x_j \in X$ are sampled states and $f(x_i), f(x_j)$ are the corresponding observable values, then the sampled slopes are

$$\frac{|f(x_i) - f(x_j)|}{d_X(x_i, x_j)},$$

where d_X is the metric of the ambient space. The code records both the maximum sampled slope and the 0.99-quantile of these slopes. Dividing the empirical partial diameter by these sampled Lipschitz constants gives two normalized proxies:

$$\frac{\operatorname{diam}(f_*\mu_X; 1-\kappa)}{L_{\max}} \quad \text{and} \quad \frac{\operatorname{diam}(f_*\mu_X; 1-\kappa)}{L_{0.99}}.$$

If the chosen observable were exactly 1-Lipschitz, these normalized quantities would coincide with the raw width. In practice they should be viewed only as heuristic lower-scale corrections.

1.2.1 Random sampling using standard uniform measure on the unit sphere

The first family of spaces is the real unit sphere

$$S^{m-1} = \{x = (x_1, \dots, x_m) \in \mathbb{R}^m : \|x\|_2 = 1\},$$

equipped with the geodesic distance

$$d_S(x, y) = \arccos\langle x, y\rangle$$

and the normalized Riemannian volume measure. This is the standard metric-measure structure used in concentration of measure on spheres [Lee18; Ver18; Shi14].

Sampling is performed by drawing a standard Gaussian vector $g \in \mathbb{R}^m$ and normalizing:

$$x = \frac{g}{\|g\|_2}.$$

This produces the uniform distribution on S^{m-1}.

The observable is a Shannon entropy built from the squared coordinates:

$$f_{\text{sphere}}(x) = -\sum_{i=1}^{m} x_i^2 \log_2(x_i^2).$$

Since $(x_1^2, \dots, x_m^2)$ is a probability vector, f_{sphere} takes values in $[0, \log_2 m]$, and the code records $\log_2 m$ as the natural upper bound of the observable.

For each chosen dimension m, the experiment generates N independent samples $x^{(1)}, \dots, x^{(N)}$, computes the values

$$f_{\text{sphere}}(x^{(1)}), \dots, f_{\text{sphere}}(x^{(N)}),$$

and then evaluates the shortest interval containing mass at least $1 - \kappa$. This gives an empirical observable-diameter proxy for the sphere family. The code also computes the empirical mean, median, standard deviation, and the normalized proxies obtained from sampled Lipschitz ratios.

1.2.2 Visualized the concentration of measure phenomenon on complex projective space

The second family is complex projective space

$$\mathbb{C}P^{d_A d_B - 1},$$

viewed as the space of pure states in $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ modulo global phase. Geometrically, this space is equipped with the Fubini–Study metric and its associated normalized volume measure [Lee18; BZ17]. Numerically, a projective point is represented by a unit vector

$$\psi \in \mathbb{C}^{d_A d_B}, \qquad \|\psi\| = 1,$$

and distances are computed by the Fubini–Study formula

$$d_{FS}([\psi], [\phi]) = \arccos |\langle \psi, \phi \rangle|.$$

Sampling is implemented by drawing a complex Gaussian matrix

$$G \in \mathbb{C}^{d_A \times d_B},$$

with independent standard complex normal entries, and then normalizing it so that

$$\psi = \frac{\operatorname{vec}(G)}{\|\operatorname{vec}(G)\|}.$$

This is equivalent to Haar sampling on the unit sphere in $\mathbb{C}^{d_A d_B}$ and hence induces the standard unitarily invariant measure on $\mathbb{C}P^{d_A d_B - 1}$ [BZ17; NC10].

The real-valued observable is the bipartite entanglement entropy. Writing

$$\rho_A = \operatorname{Tr}_B |\psi\rangle\langle\psi|,$$

the code defines

$$f_{\mathrm{CP}}([\psi]) = S(\rho_A) = -\operatorname{Tr}(\rho_A \log_2 \rho_A).$$

Equivalently, if $\lambda_1, \ldots, \lambda_{d_A}$ are the eigenvalues of ρ_A, then

$$f_{\mathrm{CP}}([\psi]) = -\sum_{i=1}^{d_A} \lambda_i \log_2 \lambda_i.$$

This observable takes values in $[0, \log_2 d_A]$.

For each dimension pair (d_A, d_B), the experiment samples N independent Haar-random pure states, computes the entropy values, and then forms the empirical push-forward distribution on $\mathbb{R}$. The shortest interval containing mass at least $1-\kappa$ is reported as the entropy-based observable-diameter proxy. In addition, the code plots histograms, upper-tail deficit plots for

$$\log_2 d_A - S(\rho_A),$$

and family-wise comparisons of partial diameter, standard deviation, and mean deficit. When available, these plots are overlaid with the Page average entropy and with Hayden-style concentration scales, which serve as theoretical guides rather than direct outputs of the simulation [Hay10; HLW06; San95].

1.2.3 Random sampling using Majorana Stellar representation

The third family is the symmetric subspace

$$\mathrm{Sym}^N(\mathbb{C}^2),$$

which is naturally identified with $\mathbb{C}P^N$ after projectivization. In this model, a pure symmetric N-qubit state is written in the Dicke basis as

$$|\psi\rangle = \sum_{k=0}^{N} c_k |D_k^N\rangle, \qquad \sum_{k=0}^{N} |c_k|^2 = 1.$$

The projective metric is again the Fubini–Study metric

$$d_{FS}([\psi],[\phi]) = \arccos |\langle \psi, \phi\rangle|.$$

Sampling is performed by drawing a standard complex Gaussian vector

$$(c_0, \ldots, c_N) \in \mathbb{C}^{N+1}$$

and normalizing it. This gives the unitarily invariant measure on the projective symmetric state space.

The observable used by the code is the one-particle entropy of the symmetric state. From the coefficient vector $(c_0, \ldots, c_N)$ one constructs the one-qubit reduced density matrix ρ_1, and then defines

$$f_{\mathrm{Maj}}([\psi]) = S(\rho_1) = -\operatorname{Tr}(\rho_1 \log_2 \rho_1).$$

Since ρ_1 is a qubit state, this observable takes values in $[0, 1]$.

To visualize the same states in Majorana form, the code also associates to a sampled symmetric state its Majorana polynomial and computes its roots. After stereographic projection, these roots define N points on S^2, called the Majorana stars [BZ17]. The resulting star plots are included only as geometric visualizations; they are not used to define the metric or the observable. The metric-measure structure used in the actual simulation remains the Fubini–Study metric and the unitarily invariant measure on the projective symmetric state space.

Thus, for each N, the simulation produces:

1. a sample of symmetric states,
2. the corresponding one-body entropy values,
3. the shortest interval containing mass at least $1 - \kappa$ in the push-forward distribution on $\mathbb{R}$,
4. empirical Lipschitz-normalized versions of this width,
5. and a separate Majorana-star visualization of representative samples.

Taken together, these three families allow us to compare how entropy-based concentration behaves on a real sphere, on a general complex projective space carrying bipartite entanglement entropy, and on the symmetric subspace described by Majorana stellar data.

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