

# Chapter 1

## Levy's family and observable diameters

### Abstract

The concentration of measure phenomenon comes with an important notion called the observable diameter, which measures how tightly every 1-Lipschitz function  $f : X \rightarrow \mathbb{R}$  concentrates on a metric-measure space  $X$  outside a set of measure at most  $\kappa$ .

From Hayden's work, we know that for a random pure state in a bipartite system the entropy is nearly maximal and the entropy function has Lipschitz constant upper bounded by  $\sqrt{8} \log_2(d_A)$  for  $d_A \geq 3$ . We can use the entropy function as a proxy for the observable diameter.

Altogether, we study how entropy concentration reflects the geometry of high-dimensional spheres, generic bipartite projective state spaces, and symmetric quantum state manifolds.

In this section, we will explore how the results from Hayden's concentration of measure theorem can be understood in terms of observable diameters from Gromov's perspective and what properties they reveal for entropy functions.

We will try to use the results from previous sections to estimate the observable diameter for complex projective spaces.

### 1.1 Observable diameters

This chapter starts from the qualitative concentration statements in Chapter 1 and asks for a geometric quantity that records the same phenomenon directly at the level of metric-measure spaces. Observable diameter is that quantity. It translates concentration of real observables into an invariant of the underlying space, which makes it possible to compare spheres and complex projective spaces on the same footing.

Recall from previous sections, an arbitrary 1-Lipschitz function  $f : S^n \rightarrow \mathbb{R}$  concentrates near a single value  $a_0 \in \mathbb{R}$  as strongly as the distance function does.

**Definition 1.** Let  $X$  be a topological space with the following:

1.  $X$  is complete.
2.  $X$  is a metric space with metric  $d_X$ .
3.  $X$  has a Borel probability measure  $\mu_X$ .

Then  $(X, d_X, \mu_X)$  is a **metric measure space**.

**Definition 2.** Let  $(X, d_X)$  be a metric space. The **diameter** of a set  $A \subset X$  is defined as

$$\text{diam}(A) = \sup_{x, y \in A} d_X(x, y).$$

**Definition 3.** Let  $(X, d_X, \mu_X)$  be a metric measure space. For any real number  $\alpha \leq 1$ , the **partial diameter** of  $X$  is defined as

$$\text{diam}(X; \alpha) = \inf_{A \subset X, \mu_X(A) \geq \alpha} \text{diam}(A).$$

This definition generalizes the relation between the measure and metric in a metric-measure space. Intuitively, a space with smaller partial diameter can carry more mass inside the same diameter constraint.

However, in higher dimensions, the volume may tend to concentrate more around a small neighborhood of a set, as we saw earlier for high-dimensional spheres. We can safely cut away  $\kappa > 0$  mass to reduce the diameter significantly, and this yields a better measure of concentration.

**Definition 4.** Let  $X$  be a metric-measure space,  $Y$  be a metric space, and  $f : X \rightarrow Y$  be a 1-Lipschitz function. Then  $f_*\mu_X = \mu_Y$  is a push-forward measure on  $Y$ .

For any real number  $\kappa > 0$ , the  **$\kappa$ -observable diameter with screen  $Y$**  is defined as

$$\text{ObserDiam}_Y(X; -\kappa) = \sup\{\text{diam}(f_*\mu_X; 1 - \kappa) \mid f : X \rightarrow Y \text{ is 1-Lipschitz}\}.$$

And the **observable diameter with screen  $Y$**  is defined as

$$\text{ObserDiam}_Y(X) = \inf_{\kappa > 0} \max\{\text{ObserDiam}_Y(X; -\kappa), \kappa\}.$$

If  $Y = \mathbb{R}$ , we call it the **observable diameter**.

If we collapse it naively via

$$\inf_{\kappa > 0} \text{ObserDiam}_Y(X; -\kappa),$$

we typically get something degenerate: as  $\kappa \rightarrow 1$ , the condition “mass  $\geq 1 - \kappa$ ” becomes almost empty, so  $\text{diam}(\nu; 1 - \kappa)$  can be forced to be 0 by taking a tiny set of positive mass. Hence the infimum tends to 0 for essentially any non-atomic space.

This is why one either:

1. keeps  $\text{ObserDiam}_Y(X; -\kappa)$  as a *function of  $\kappa$*  (picking  $\kappa$  to be small but not 0), or

2. if one insists on a single number, balances “spread” against “exceptional mass” by defining  $\text{ObserDiam}_Y(X) = \inf_{\kappa > 0} \max\{\text{ObserDiam}_Y(X; -\kappa), \kappa\}$  as above.

The point of the  $\max\{\cdot, \kappa\}$  is that it prevents cheating by taking  $\kappa$  close to 1: if  $\kappa$  is large then the maximum is large regardless of how small  $\text{ObserDiam}_Y(X; -\kappa)$  is, so the infimum is forced to occur where the exceptional mass and the observable spread are small.

Few additional propositions in [Shi14] will help us estimate the observable diameter for complex projective spaces.

**Proposition 5.** *Let  $X$  and  $Y$  be two metric-measure spaces and  $\kappa > 0$ , and let  $f : Y \rightarrow X$  be a 1-Lipschitz function ( $Y$  dominates  $X$ , denoted as  $X \prec Y$ ). Then:*

1.  $\text{diam}(X; 1 - \kappa) \leq \text{diam}(Y; 1 - \kappa)$
2.  $\text{ObserDiam}(X; -\kappa) \leq \text{diam}(X; 1 - \kappa)$ , and  $\text{ObserDiam}(X)$  is finite.
3.  $\text{ObserDiam}(X; -\kappa) \leq \text{ObserDiam}(Y; -\kappa)$

*Proof.* Since  $f$  is 1-Lipschitz, we have  $f_*\mu_Y = \mu_X$ . Let  $A$  be any Borel set of  $Y$  with  $\mu_Y(A) \geq 1 - \kappa$  and  $\overline{f(A)}$  be the closure of  $f(A)$  in  $X$ . We have  $\mu_X(\overline{f(A)}) = \mu_Y(f^{-1}(\overline{f(A)})) \geq \mu_Y(A) \geq 1 - \kappa$  and by the 1-Lipschitz property,  $\text{diam}(\overline{f(A)}) \leq \text{diam}(A)$ , so  $\text{diam}(X; 1 - \kappa) \leq \text{diam}(A) \leq \text{diam}(Y; 1 - \kappa)$ .

Let  $g : X \rightarrow \mathbb{R}$  be any 1-Lipschitz function. Since  $(\mathbb{R}, |\cdot|, g_*\mu_X)$  is dominated by  $X$ ,  $\text{diam}(\mathbb{R}; 1 - \kappa) \leq \text{diam}(X; 1 - \kappa)$ . Therefore,  $\text{ObserDiam}(X; -\kappa) \leq \text{diam}(X; 1 - \kappa)$ .

And

$$\text{diam}(g_*\mu_X; 1 - \kappa) \leq \text{diam}((g \circ f)_*\mu_Y; 1 - \kappa) \leq \text{ObserDiam}(Y; -\kappa).$$

□

**Proposition 6.** *Let  $X$  be a metric-measure space. Then for any real number  $t > 0$ , we have*

$$\text{ObserDiam}(tX; -\kappa) = t \text{ObserDiam}(X; -\kappa)$$

where  $tX = (X, td_X, \mu_X)$ .

*Proof.*

$$\begin{aligned} \text{ObserDiam}(tX; -\kappa) &= \sup\{\text{diam}(f_*\mu_X; 1 - \kappa) \mid f : tX \rightarrow \mathbb{R} \text{ is 1-Lipschitz}\} \\ &= \sup\{\text{diam}(f_*\mu_X; 1 - \kappa) \mid t^{-1}f : X \rightarrow \mathbb{R} \text{ is 1-Lipschitz}\} \\ &= \sup\{\text{diam}((tg)_*\mu_X; 1 - \kappa) \mid g : X \rightarrow \mathbb{R} \text{ is 1-Lipschitz}\} \\ &= t \sup\{\text{diam}(g_*\mu_X; 1 - \kappa) \mid g : X \rightarrow \mathbb{R} \text{ is 1-Lipschitz}\} \\ &= t \text{ObserDiam}(X; -\kappa). \end{aligned}$$

□

### 1.1.1 Observable diameter for class of spheres

With the basic formalism in place, the next step is to test it on the standard family where concentration is best understood: high-dimensional spheres. This is the model case in which Gaussian limits and scaling arguments can be made explicit, and it provides the comparison space that will later control complex projective space through the Hopf fibration.

In this section, we will try to use the results from previous sections to estimate the observable diameter for the class of spheres.

**Theorem 7.** *For any real number  $\kappa$  with  $0 < \kappa < 1$ , we have*

$$\text{ObsDiam}(S^n(1); -\kappa) = O\left(\frac{1}{\sqrt{n}}\right).$$

*Proof.* First, recall that by the Maxwell-Boltzmann distribution law, for any  $n > 0$ , if  $I(r)$  denotes the measure of the standard Gaussian measure on the interval  $[0, r]$ , then we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \text{ObsDiam}(S^n(\sqrt{n}); -\kappa) &= \lim_{n \rightarrow \infty} \sup\{\text{diam}((\pi_{n,k})_* \sigma^n; 1 - \kappa) \mid \pi_{n,k} \text{ is 1-Lipschitz}\} \\ &= \lim_{n \rightarrow \infty} \sup\{\text{diam}(\gamma^1; 1 - \kappa) \mid \gamma^1 \text{ is the standard Gaussian measure}\} \\ &= \text{diam}(\gamma^1; 1 - \kappa) \\ &= 2I^{-1}\left(\frac{1 - \kappa}{2}\right). \end{aligned}$$

By Proposition 6, we have

$$\text{ObsDiam}(S^n(\sqrt{n}); -\kappa) = \sqrt{n} \text{ObsDiam}(S^n(1); -\kappa).$$

So

$$\text{ObsDiam}(S^n(1); -\kappa) = \frac{1}{\sqrt{n}} 2I^{-1}\left(\frac{1 - \kappa}{2}\right) = O\left(\frac{1}{\sqrt{n}}\right).$$

□

From the previous discussion, we see that the only remaining step in finding the observable diameter of  $\mathbb{C}P^n$  is to identify a 1-Lipschitz map with the correct push-forward measure.

To find such a metric, we need some additional results.

**Definition 8.** *Let  $M$  be a smooth manifold. A **Riemannian metric** on  $M$  is a smooth covariant tensor field  $g \in \mathcal{T}^2(M)$  such that for each  $p \in M$ ,  $g_p$  is an inner product on  $T_p M$ .*

$g_p(v, v) \geq 0$  for each  $p \in M$  and each  $v \in T_p M$ . Equality holds if and only if  $v = 0$ .

**Theorem 9.** *Let  $(\tilde{M}, \tilde{g})$  be a Riemannian manifold, let  $\pi : \tilde{M} \rightarrow M$  be a surjective smooth submersion, and let  $G$  be a group acting on  $\tilde{M}$ . If the **action** is*

1. *isometric: the map  $x \mapsto \varphi \cdot x$  is an isometry for each  $\varphi \in G$ .*
2. *vertical: every element  $\varphi \in G$  takes each fiber to itself, that is  $\pi(\varphi \cdot p) = \pi(p)$  for all  $p \in \tilde{M}$ .*

3. *transitive on fibers: for each  $p, q \in \tilde{M}$  such that  $\pi(p) = \pi(q)$ , there exists  $\varphi \in G$  such that  $\varphi \cdot p = q$ .*

Then there is a unique Riemannian metric on  $M$  such that  $\pi$  is a Riemannian submersion.

A natural measure for  $\mathbb{C}P^n$  is the normalized volume measure on  $\mathbb{C}P^n$  induced from the Fubini-Study metric [Lee18, Example 2.30].

**Definition 10.** *Let  $n$  be a positive integer, and consider the complex projective space  $\mathbb{C}P^n$  defined as the quotient space of  $\mathbb{C}^{n+1}$  by the equivalence relation  $z \sim z'$  if there exists  $\lambda \in \mathbb{C}$  such that  $z = \lambda z'$ . The map  $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{C}P^n$  sending each point in  $\mathbb{C}^{n+1} \setminus \{0\}$  to its span is a surjective smooth submersion.*

*Identifying  $\mathbb{C}^{n+1}$  with  $\mathbb{R}^{2n+2}$  with its Euclidean metric, we can view the unit sphere  $S^{2n+1}$  with its round metric  $\mathring{g}$  as an embedded Riemannian submanifold of  $\mathbb{C}^{n+1} \setminus \{0\}$ . Let  $p : S^{2n+1} \rightarrow \mathbb{C}P^n$  denote the restriction of the map  $\pi$ . Then  $p$  is smooth, and it is surjective, because every 1-dimensional complex subspace contains elements of unit norm.*

There are many additional properties for such a construction, and here we check only the point needed later.

We need to show that it is a submersion.

*Proof.* Let  $z_0 \in S^{2n+1}$  and set  $\zeta_0 = p(z_0) \in \mathbb{C}P^n$ . Since  $\pi$  is a smooth submersion, it has a smooth local section  $\sigma : U \rightarrow \mathbb{C}^{n+1}$  defined on a neighborhood  $U$  of  $\zeta_0$  and satisfying  $\sigma(\zeta_0) = z_0$  by the local section theorem (Theorem ??). Let  $v : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow S^{2n+1}$  be the radial projection onto the sphere:

$$v(z) = \frac{z}{|z|}.$$

Since dividing an element of  $\mathbb{C}^{n+1}$  by a nonzero scalar does not change its span, it follows that  $p \circ v = \pi$ . Therefore, if we set  $\tilde{\sigma} = v \circ \sigma$ , then  $\tilde{\sigma}$  is a smooth local section of  $p$ . Applying the local section theorem again shows that  $p$  is a submersion.

Define an action of  $S^1$  on  $S^{2n+1}$  by complex multiplication:

$$\lambda(z^1, z^2, \dots, z^{n+1}) = (\lambda z^1, \lambda z^2, \dots, \lambda z^{n+1})$$

for  $\lambda \in S^1$  (viewed as a complex number of norm 1) and  $z = (z^1, z^2, \dots, z^{n+1}) \in S^{2n+1}$ . This action is isometric, vertical, and transitive on fibers of  $p$ .

By Theorem 9, there is a unique metric on  $\mathbb{C}P^n$  such that the map  $p : S^{2n+1} \rightarrow \mathbb{C}P^n$  is a Riemannian submersion. This metric is called the Fubini–Study metric.  $\square$

### 1.1.2 Observable diameter for complex projective spaces

The sphere estimate by itself is not yet a statement about quantum state space. The missing geometric bridge is the Hopf fibration, which realizes complex projective space as a quotient of the

sphere by phase. Because this quotient map is a Riemannian submersion, the concentration scale on the sphere can be transferred to  $\mathbb{C}P^n$ .

Using the projection map and Hopf's fibration, we can estimate the observable diameter for complex projective spaces from the observable diameter of spheres.

**Theorem 11.** *For any real number  $\kappa$  with  $0 < \kappa < 1$ , we have*

$$\text{ObsDiam}(\mathbb{C}P^n(1); -\kappa) \leq O\left(\frac{1}{\sqrt{n}}\right).$$

*Proof.* Recall from Example 2.30 in [Lee18] that the Hopf fibration  $f_n : S^{2n+1}(1) \rightarrow \mathbb{C}P^n$  is 1-Lipschitz with respect to the Fubini-Study metric on  $\mathbb{C}P^n$ . The push-forward  $(f_n)_*\sigma^{2n+1}$  coincides with the normalized volume measure on  $\mathbb{C}P^n$  induced from the Fubini-Study metric.

By Proposition 5, we have

$$\text{ObsDiam}(\mathbb{C}P^n(1); -\kappa) \leq \text{ObsDiam}(S^{2n+1}(1); -\kappa) \leq O\left(\frac{1}{\sqrt{n}}\right).$$

□

## 1.2 Use entropy function as estimator of observable diameter for complex projective spaces

The previous subsection gives an abstract upper bound. The next question is how to test that scale numerically on observables that arise naturally in quantum information. Entropy is the obvious candidate: it is physically meaningful, it is already controlled by Hayden's theorem, and after normalization by a Lipschitz constant it supplies a computable lower-bound proxy for observable diameter.

In this section we describe a Monte Carlo pipeline for comparing concentration phenomena across three metric-measure spaces using real-valued entropy observables. The goal is not to compute the exact observable diameter

$$\text{ObsDiam}_{\mathbb{R}}(X; -\kappa) = \sup_{f \in \text{Lip}_1(X, \mathbb{R})} \text{diam}(f_*\mu_X; 1 - \kappa),$$

but to estimate it by choosing a specific observable  $f : X \rightarrow \mathbb{R}$  and then measuring the partial diameter of its push-forward distribution. Thus all numerical quantities below should be interpreted as *entropy-based observable-diameter proxies*, not exact observable diameters in Gromov's sense [Gro81; Shi14].

The screen is  $\mathbb{R}$  equipped with the Euclidean metric, and for a fixed  $\kappa \in (0, 1)$  we set

$$\alpha = 1 - \kappa.$$

Given sampled values  $y_1, \dots, y_N \in \mathbb{R}$  of the observable, the code sorts them and computes the shortest interval  $[a, b]$  containing at least  $\lceil \alpha N \rceil$  samples. Its width

$$b - a$$

is the empirical partial diameter of the push-forward measure on  $\mathbb{R}$ .

To compare this width with the true observable diameter, the code also estimates an empirical Lipschitz constant of the chosen observable. If  $x_i, x_j \in X$  are sampled states and  $f(x_i), f(x_j)$  are the corresponding observable values, then the sampled slopes are

$$\frac{|f(x_i) - f(x_j)|}{d_X(x_i, x_j)},$$

where  $d_X$  is the metric of the ambient space. The code records both the maximum sampled slope and the 0.99-quantile of these slopes. Dividing the empirical partial diameter by these sampled Lipschitz constants gives two normalized proxies:

$$\frac{\text{diam}(f_*\mu_X; 1 - \kappa)}{L_{\max}} \quad \text{and} \quad \frac{\text{diam}(f_*\mu_X; 1 - \kappa)}{L_{0.99}}.$$

If the chosen observable were exactly 1-Lipschitz, these normalized quantities would coincide with the raw width. In practice they should be viewed only as heuristic lower-scale corrections.

### 1.2.1 Random sampling using standard uniform measure on the unit sphere

The first family of spaces is the real unit sphere

$$S^{m-1} = \{x = (x_1, \dots, x_m) \in \mathbb{R}^m : \|x\|_2 = 1\},$$

equipped with the geodesic distance

$$d_S(x, y) = \arccos\langle x, y \rangle$$

and the normalized Riemannian volume measure. This is the standard metric-measure structure used in concentration of measure on spheres [Lee18; Ver18; Shi14].

Sampling is performed by drawing a standard Gaussian vector  $g \in \mathbb{R}^m$  and normalizing:

$$x = \frac{g}{\|g\|_2}.$$

This produces the uniform distribution on  $S^{m-1}$ .

The observable is a Shannon entropy built from the squared coordinates:

$$f_{\text{sphere}}(x) = - \sum_{i=1}^m x_i^2 \log_2(x_i^2).$$

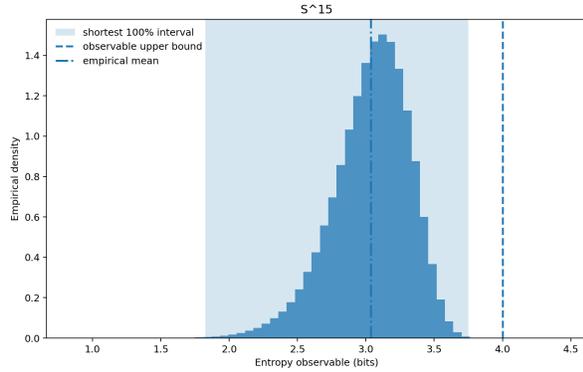
Since  $(x_1^2, \dots, x_m^2)$  is a probability vector,  $f_{\text{sphere}}$  takes values in  $[0, \log_2 m]$ , and the code records  $\log_2 m$  as the natural upper bound of the observable.

For each chosen dimension  $m$ , the experiment generates  $N$  independent samples  $x^{(1)}, \dots, x^{(N)}$ , computes the values

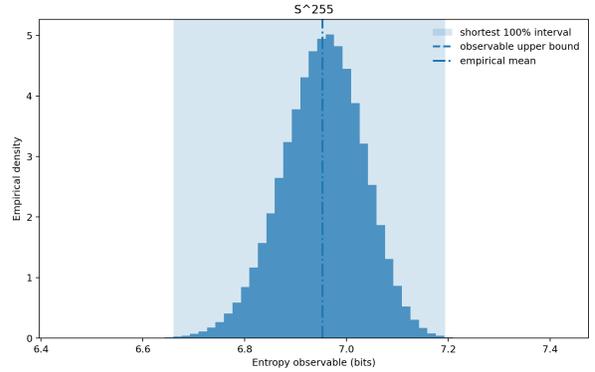
$$f_{\text{sphere}}(x^{(1)}), \dots, f_{\text{sphere}}(x^{(N)}),$$

and then evaluates the shortest interval containing mass at least  $1 - \kappa$ . This gives an empirical observable-diameter proxy for the sphere family. The code also computes the empirical mean, median, standard deviation, and the normalized proxies obtained from sampled Lipschitz ratios.

The experiment produces histograms of the observable values, upper-tail deficit plots for  $\log_2 m - f_{\text{sphere}}(x)$ , and family-wise comparisons of partial diameter, standard deviation, and mean deficit. When available, these plots are overlaid with theoretical concentration scales derived from Lévy’s lemma and related results [Lee18].



Entropy distribution for  $S^{15}$



Entropy distribution for  $S^{255}$

### 1.2.2 Visualized the concentration of measure phenomenon on complex projective space

The second family is complex projective space

$$\mathbb{C}P^{d_A d_B - 1},$$

viewed as the space of pure states in  $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$  modulo global phase. Geometrically, this space is equipped with the Fubini–Study metric and its associated normalized volume measure [Lee18; BZ17]. Numerically, a projective point is represented by a unit vector

$$\psi \in \mathbb{C}^{d_A d_B}, \quad \|\psi\| = 1,$$

and distances are computed by the Fubini–Study formula

$$d_{FS}([\psi], [\phi]) = \arccos |\langle \psi, \phi \rangle|.$$

Sampling is implemented by drawing a complex Gaussian matrix

$$G \in \mathbb{C}^{d_A \times d_B},$$

with independent standard complex normal entries, and then normalizing it so that

$$\psi = \frac{\text{vec}(G)}{\|\text{vec}(G)\|}.$$

This is equivalent to Haar sampling on the unit sphere in  $\mathbb{C}^{d_A d_B}$  and hence induces the standard unitarily invariant measure on  $\mathbb{C}P^{d_A d_B - 1}$  [BZ17; NC10].

The real-valued observable is the bipartite entanglement entropy. Writing

$$\rho_A = \text{Tr}_B |\psi\rangle\langle\psi|,$$

the code defines

$$f_{\text{CP}}([\psi]) = S(\rho_A) = -\text{Tr}(\rho_A \log_2 \rho_A).$$

Equivalently, if  $\lambda_1, \dots, \lambda_{d_A}$  are the eigenvalues of  $\rho_A$ , then

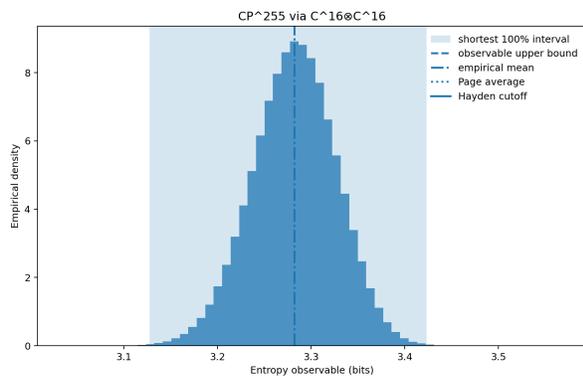
$$f_{\text{CP}}([\psi]) = -\sum_{i=1}^{d_A} \lambda_i \log_2 \lambda_i.$$

This observable takes values in  $[0, \log_2 d_A]$ .

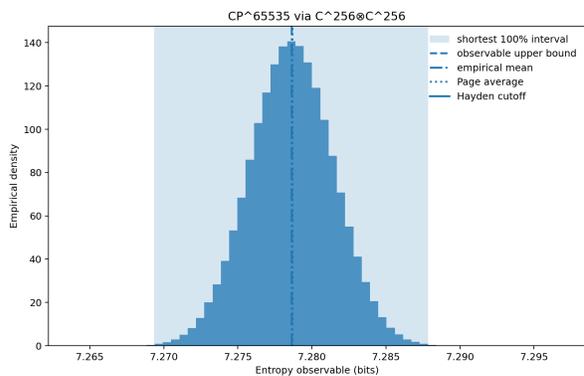
For each dimension pair  $(d_A, d_B)$ , the experiment samples  $N$  independent Haar-random pure states, computes the entropy values, and then forms the empirical push-forward distribution on  $\mathbb{R}$ . The shortest interval containing mass at least  $1 - \kappa$  is reported as the entropy-based observable-diameter proxy. In addition, the code plots histograms, upper-tail deficit plots for

$$\log_2 d_A - S(\rho_A),$$

and family-wise comparisons of partial diameter, standard deviation, and mean deficit. When available, these plots are overlaid with the Page average entropy and with Hayden-style concentration scales, which serve as theoretical guides rather than direct outputs of the simulation [Hay10; HLW06; San95].



Entropy distribution for  $(d_A, d_B) = (16, 16)$



Entropy distribution for  $(d_A, d_B) = (256, 256)$

### 1.3 A conjecture on observable diameter for complex projective spaces

The numerical section does not compute  $\text{ObserDiam}(\mathbb{C}P^n(1); -\kappa)$  exactly, but it does produce a natural lower-bound proxy. If  $f : \mathbb{C}P^n \rightarrow \mathbb{R}$  has Lipschitz constant  $L_f > 0$ , then  $L_f^{-1}f$  is 1-Lipschitz, so

$$\frac{\text{diam}(f_*\mu; 1 - \kappa)}{L_f} \leq \text{ObserDiam}(\mathbb{C}P^n(1); -\kappa).$$

Taking  $f$  to be the entropy observable means that the normalized entropy widths from the simulations cannot determine the full observable diameter, but they do give a computable lower bound on its scale. In that sense the entropy function is a probe of projective-space concentration, and the conjecture below asks that the upper bound coming from the Hopf-fibration argument has the same order as the concentration already suggested by entropy.

**Theorem 12** (Wu’s conjecture). *For  $0 < \kappa < 1$ ,*

$$\text{ObserDiam}(\mathbb{C}P^n(1); -\kappa) = O\left(\frac{1}{\sqrt{n}}\right).$$

**Sketch of the proof.** The expected upper bound should come from the same geometric mechanism as on the sphere. The Hopf fibration  $S^{2n+1}(1) \rightarrow \mathbb{C}P^n(1)$  is 1-Lipschitz and sends the normalized spherical measure to the normalized Fubini–Study volume measure, so domination already gives

$$\text{ObserDiam}(\mathbb{C}P^n(1); -\kappa) \leq \text{ObserDiam}(S^{2n+1}(1); -\kappa).$$

Thus the sphere estimate suggests the order  $n^{-1/2}$  on projective space as well. To compare this upper bound with entropy data, one studies the entropy observable  $f([\psi]) = S(\text{Tr}_B |\psi\rangle\langle\psi|)$  on the relevant projective state space, normalizes by a Lipschitz bound for  $f$ , and uses the resulting widths as lower-bound proxies for observable diameter. The remaining work is to relate these entropy-based lower bounds to intrinsic geometric concentration, possibly through curvature methods such as Fubini–Study geometry and Lévy–Gromov type inequalities.

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