Chapter 1

Levy's family and observable diameters

In this section, we will explore how the results from Hayden's concentration of measure theorem can be understood in terms of observable diameters from Gromov's perspective and what properties it reveals for entropy functions.

We will try to use the results from previous sections to estimate the observable diameter for complex projective spaces.

1.1 Observable diameters

Recall from previous sections, an arbitrary 1-Lipschitz function $f : S^n \to \mathbb{R}$ concentrates near a single value $a_0 \in \mathbb{R}$ as strongly as the distance function does.

Definition 1. *Let X be a topological space with the following:*

1. *X is a complete (every Cauchy sequence converges)*
2. *X is a metric space with metric d_X*
3. *X has a Borel probability measure μ_X*

Then (X, d_X, μ_X) is a ***metric measure space****.*

Definition 2. *Let (X, d_X) be a metric space. The* ***diameter*** *of a set $A \subset X$ is defined as*

$$\operatorname{diam}(A) = \sup_{x,y\in A} d_X(x, y).$$

Definition 3. *Let (X, d_X, μ_X) be a metric measure space, For any real number $\alpha \leq 1$, the* ***partial diameter*** *of X is defined as*

$$\operatorname{diam}(A; \alpha) = \inf_{A\subseteq X|\mu_X(A)\geq\alpha} \operatorname{diam}(A).$$

This definition generalize the relation between the measure and metric in the metric-measure space. Intuitively, the space with smaller partial diameter can take more volume with the same diameter constrains.

However, in higher dimensions, the volume may tend to concentrates more around a small neighborhood of the set, as we see in previous chapters with high dimensional sphere as example. We can safely cut $\kappa > 0$ volume to significantly reduce the diameter, this yields better measure for concentration for shapes in spaces with high dimension.

Definition 4. *Let X be a metric-measure space, Y be a metric space, and $f : X \to Y$ be a 1-Lipschitz function. Then $f_*\mu_X = \mu_Y$ is a push forward measure on Y.*

*For any real number $\kappa > 0$, the κ-**observable diameter with screen** Y is defined as*

$$\mathrm{ObserDiam}_Y(X;\kappa) = \sup\{\mathrm{diam}(f_*\mu_X; 1-\kappa)\}$$

*And the **obbservable diameter with screen** Y is defined as*

$$\mathrm{ObserDiam}_Y(X) = \inf_{\kappa>0} \max\{\mathrm{ObserDiam}_Y(X;\kappa)\}$$

*If $Y = \mathbb{R}$, we call it the **observable diameter**.*

If we collapse it naively via

$$\inf_{\kappa>0} \mathrm{ObserDiam}_Y(X;\kappa),$$

we typically get something degenerate: as $\kappa \to 1$, the condition "mass $\geq 1-\kappa$" becomes almost empty space, so $\mathrm{diam}(\nu; 1-\kappa)$ can be forced to be 0 (take a tiny set of positive mass), and hence the infimum tends to 0 for essentially any non-atomic space.

This is why one either:

1. keeps $\mathrm{ObserDiam}_Y(X;\kappa)$ as a *function of* κ (picking κ to be small but not 0), or
2. if one insists on a single number, balances "spread" against "exceptional mass" by defining $\mathrm{ObserDiam}_Y(X) = \inf_{\kappa>0} \max\{\mathrm{ObserDiam}_Y(X;\kappa), \kappa\}$ as above.

The point of the $\max\{\cdot, \kappa\}$ is that it prevents cheating by taking κ close to 1: if κ is large then the maximum is large regardless of how small $\mathrm{ObserDiam}_Y(X;\kappa)$ is, so the infimum is forced to occur where the exceptional mass and the observable spread are small.

Few additional proposition in [Shi14] will help us to estimate the observable diameter for complex projective spaces.

Proposition 5. *Let X and Y be two metric-measure spaces and $\kappa > 0$, and let $f : Y \to X$ be a 1-Lipschitz function (Y dominates X, denoted as $X \prec Y$) then:*

1. $$\mathrm{diam}(X, 1-\kappa) \leq \mathrm{diam}(Y, 1-\kappa)$$
2. $\mathrm{ObserDiam}(X;-\kappa) \leq \mathrm{diam}(X;1-\kappa)$*, and* $\mathrm{ObserDiam}(X)$ *is finite.*
3. $$\mathrm{ObserDiam}(X;-\kappa) \leq \mathrm{ObserDiam}(Y;-\kappa)$$

Proof. Since f is 1-Lipschitz, we have $f_*\mu Y = \mu_X$. Let A be any Borel set of Y with $\mu_Y(A) \geq 1-\kappa$ and $\overline{f(A)}$ be the closure of $f(A)$ in X. We have $\mu_X(\overline{f(A)}) = \mu_Y(f^{-1}(\overline{f(A)})) \geq \mu_Y(A) \geq 1-\kappa$ and by the 1-lipschitz property, $\text{diam}(\overline{f(A)}) \leq \text{diam}(A)$, so $\text{diam}(X; 1-\kappa) \leq \text{diam}(A) \leq \text{diam}(Y; 1-\kappa)$.

Let $g : X \to \mathbb{R}$ be any 1-lipschitz function, since $(\mathbb{R}, |\cdot|, g_*\mu_X)$ is dominated by X, $\text{diam}(\mathbb{R}; 1-\kappa) \leq \text{diam}(X; 1-\kappa)$. Therefore, $\text{ObserDiam}(X; -\kappa) \leq \text{diam}(X; 1-\kappa)$.

and

$$\text{diam}(g_*\mu_X; 1-\kappa) \leq \text{diam}((f \circ g)_*\mu_Y; 1-\kappa) \leq \text{ObserDiam}(Y; 1-\kappa)$$

□

Proposition 6. *Let X be an metric-measure space. Then for any real number $t > 0$, we have*

$$\text{ObserDiam}(tX; -\kappa) = t\,\text{ObserDiam}(X; -\kappa)$$

Where $tX = (X, tdX, \mu X)$.

Proof.

$$\begin{aligned}
\text{ObserDiam}(tX; -\kappa) &= \sup\{\text{diam}(f_*\mu_X; 1-\kappa) | f : tX \to \mathbb{R} \text{ is 1-Lipschitz}\} \\
&= \sup\{\text{diam}(f_*\mu_X; 1-\kappa) | t^{-1}f : X \to \mathbb{R} \text{ is 1-Lipschitz}\} \\
&= \sup\{\text{diam}((tg)_*\mu_X; 1-\kappa) | g : X \to \mathbb{R} \text{ is 1-Lipschitz}\} \\
&= t\sup\{\text{diam}(g_*\mu_X; 1-\kappa) | g : X \to \mathbb{R} \text{ is 1-Lipschitz}\} \\
&= t\,\text{ObserDiam}(X; -\kappa)
\end{aligned}$$

□

1.1.1 Observable diameter for class of spheres

In this section, we will try to use the results from previous sections to estimate the observable diameter for class of spheres.

Theorem 7. *For any real number κ with $0 < \kappa < 1$, we have*

$$\text{ObserDiam}(S^n(1); -\kappa) = O(\sqrt{n})$$

Proof. First, recall that by maxwell boltzmann distribution, we have that for any $n > 0$, let $I(r)$ denote the measure of standard gaussian measure on the interval $[0, r]$. Then we have

$$\begin{aligned}
\lim_{n\to\infty} \text{ObserDiam}(S^n(\sqrt{n}); -\kappa) &= \lim_{n\to\infty} \sup\{\text{diam}((\pi_{n,k})_*\sigma^n; 1-\kappa) | \pi_{n,k} \text{ is 1-Lipschitz}\} \\
&= \lim_{n\to\infty} \sup\{\text{diam}(\gamma^1; 1-\kappa) | \gamma^1 \text{ is the standard gaussian measure}\} \\
&= \text{diam}(\gamma^1; 1-\kappa) \\
&= 2I^{-1}(\frac{1-\kappa}{2}) \text{ cutting the extremum for normal distribution}
\end{aligned}$$

By proposition 6, we have

$$\mathrm{ObserDiam}(S^n(\sqrt{n}); -\kappa) = \sqrt{n}\, \mathrm{ObserDiam}(S^n(1); -\kappa)$$

So $\mathrm{ObserDiam}(S^n(1); -\kappa) = \sqrt{n}(2I^{-1}(\frac{1-\kappa}{2})) = O(\sqrt{n})$. □

1.1.2 Observable diameter for complex projective spaces

Using the projection map and Hopf's fibration, we can estimate the observable diameter for complex projective spaces from the observable diameter of spheres.

Theorem 8. *For any real number κ with $0 < \kappa < 1$, we have*

$$\mathrm{ObserDiam}(\mathbb{C}P^n(1); -\kappa) \leq O(\sqrt{n})$$

Proof. Recall from Example 2.30 in [**lee˙introduction˙2018**], the Hopf fibration $f_n : S^{2n+1}(1) \to \mathbb{C}P^n$ is 1-Lipschitz continuous with respect to the Fubini-Study metric on $\mathbb{C}P^n$. and the push-forward $(f_n)_*\sigma^{2n+1}$ coincides with the normalized volume measure on $\mathbb{C}P^n$ induced from the Fubini-Study metric.

By proposition 5, we have $\mathrm{ObserDiam}(\mathbb{C}P^n(1); -\kappa) \leq \mathrm{ObserDiam}(S^{2n+1}(1); -\kappa) \leq O(\sqrt{n})$.

□

1.1.3 More example for concentration of measure and observable diameter

In this section, we wish to use observable diameter to estimate the statics of thermal dynamics of some classical systems.

References

[Shi14] T. Shioya. *Metric measure geometry*. 2014. arXiv: 1410.0428 [math.MG]. URL: https://arxiv.org/abs/1410.0428.