Measure concentration in complex projective space and quantum entanglement

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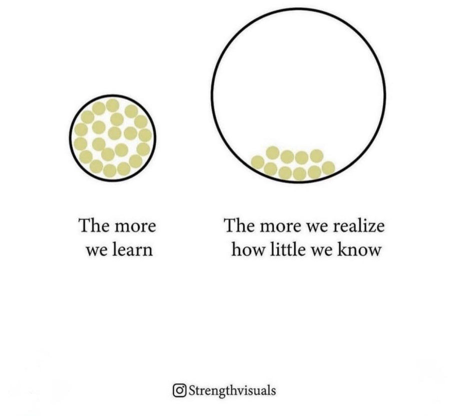
Memes



Note that the count of the beams is actually less than before.

Decomposing the statements

Concentration of measure effect

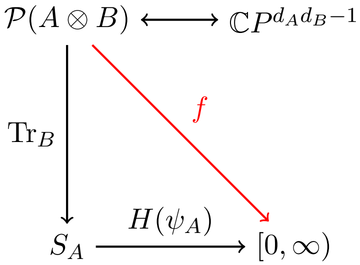
Let $\psi \in \mathcal{P}(A \otimes B)$ be a random pure state on $A \otimes B$.
If we define $\beta = \frac{1}{\ln(2)} \frac{d_A}{d_B}$, then we have

$$\Pr[H(\psi_A) < \log_2(d_A) - \alpha - \beta] \leq \exp\left(-\frac{1}{8\pi^2 \ln(2)} \frac{(d_A d_B - 1)\alpha^2}{(\log_2(d_A))^2}\right)$$

where $d_B \geq d_A \geq 3$.

[Hayden et al., 2006] Recall that the von Neumann entropy is defined as $H(\psi_A) = -\operatorname{Tr}(\psi_A \log_2(\psi_A))$.

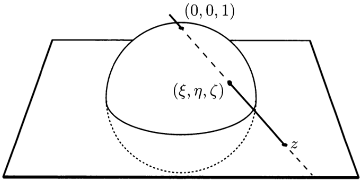
What the system actually looks like



- The red arrow is the concentration of measure effect. $f = H(\mathrm{Tr}_B(\psi))$.
- S_A denotes the mixed states on A

Wait, but what is $\mathbb{C}P^n$ and where they are coming from?

$\mathbb{C}P^n$ is the set of all complex lines in $\mathbb{C}^{n+1}$, or equivalently the space of equivalence classes of $n+1$ complex numbers up to a scalar multiple. [Bengtsson and Życzkowski, 2017]
One can find that every odd dimensional sphere S^{2n+1} under the group action of S^1, denoted by S^{2n+1}/S^1, is a complex projective space $\mathbb{C}P^n$ (complex-dimensional). Recall Math 416.



Detailed proof involves the Hopf fibration structures. It's a natural projective Hilbert space.

Some interesting claims about $\mathbb{C}P^n$

..... The claim is that every physical system can be modelled by $\mathbb{C}P^n$ for some (possibly infinite) value of n, provided taht a definite correspondence between the system and the point of $\mathbb{C}P^n$ is set up. [Bengtsson and Życzkowski, 2017]

Initial attempts for Levy's concentration lemma

Consider the orthogonal projection from $\mathbb{R}^{n+1}$, the space where S^n is embedded, to $\mathbb{R}^k$, we denote the restriction of the projection as $\pi_{n,k} : S^n(\sqrt{n}) \to \mathbb{R}^k$. Note that $\pi_{n,k}$ is a 1-Lipschitz function (projection will never increase the distance between two points).
We denote the normalized Riemannian volume measure on $S^n(\sqrt{n})$ as $\sigma^n(\cdot)$, and $\sigma^n(S^n(\sqrt{n})) = 1$.

Gaussian measure

We denote the Gaussian measure on $\mathbb{R}^k$ as γ^k.

$$d\gamma^k(x) := \frac{1}{\sqrt{2\pi}^k} \exp(-\frac{1}{2}\|x\|^2)dx$$

$x \in \mathbb{R}^k$, $\|x\|^2 = \sum_{i=1}^k x_i^2$ is the Euclidean norm, and dx is the Lebesgue measure on $\mathbb{R}^k$.

Basically, you can consider the Gaussian measure as the normalized Lebesgue measure on $\mathbb{R}^k$ with standard deviation 1.

Maxwell-Boltzmann distribution law

Maxwell-Boltzmann distribution law

For any natural number k,

$$\frac{d(\pi_{n,k})_*\sigma^n(x)}{dx} \to \frac{d\gamma^k(x)}{dx}$$

where $(\pi_{n,k})_*\sigma^n$ is the push-forward measure of σ^n by $\pi_{n,k}$.
In other words,

$$(\pi_{n,k})_*\sigma^n \to \gamma^k \text{ weakly as } n \to \infty$$

Maxwell-Boltzmann distribution law

It also has another name, the Projective limit theorem.
[Vershynin, 2018]
If $X \sim \text{Unif}(S^n(\sqrt{n}))$, then for any fixed unit vector x we have $\langle X, x \rangle \to N(0,1)$ in distribution as $n \to \infty$.

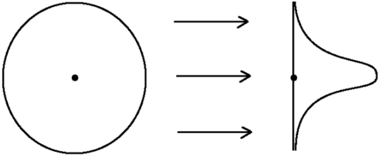


Figure 3.9 The projective central limit theorem: the projection of the uniform distribution on the sphere of radius $\sqrt{n}$ onto a line converges to the normal distribution $N(0,1)$ as $n \to \infty$.

Proof of Maxwell-Boltzmann distribution law I

This part is from [Shioya, 2014].
We denote the n dimensional volume measure on $\mathbb{R}^k$ as vol_k.
Observe that $\pi_{n,k}^{-1}(x), x \in \mathbb{R}^k$ is isometric to $S^{n-k}(\sqrt{n - \|x\|^2})$, that is, for any $x \in \mathbb{R}^k$, $\pi_{n,k}^{-1}(x)$ is a sphere with radius $\sqrt{n - \|x\|^2}$ (by the definition of $\pi_{n,k}$).
So,

$$\begin{aligned}\frac{d(\pi_{n,k})_*\sigma^n(x)}{dx} &= \frac{\mathrm{vol}_{n-k}(\pi_{n,k}^{-1}(x))}{\mathrm{vol}_k(S^n(\sqrt{n}))} \\ &= \frac{(n - \|x\|^2)^{\frac{n-k}{2}}}{\int_{\|x\| \leq \sqrt{n}} (n - \|x\|^2)^{\frac{n-k}{2}} dx}\end{aligned}$$

as $n \to \infty$.
note that $\lim_{n\to\infty} (1 - \frac{a}{n})^n = e^{-a}$ for any $a > 0$.

Proof of Maxwell-Boltzmann distribution law II

$(n-\|x\|^2)^{\frac{n-k}{2}} = \left(n(1-\frac{\|x\|^2}{n})\right)^{\frac{n-k}{2}} \to n^{\frac{n-k}{2}} \exp(-\frac{\|x\|^2}{2})$
So

$$\frac{(n-\|x\|^2)^{\frac{n-k}{2}}}{\int_{\|x\|\leq\sqrt{n}}(n-\|x\|^2)^{\frac{n-k}{2}}dx} = \frac{e^{-\frac{\|x\|^2}{2}}}{\int_{x\in\mathbb{R}^k} e^{-\frac{\|x\|^2}{2}}dx} = \frac{1}{(2\pi)^{\frac{k}{2}}}e^{-\frac{\|x\|^2}{2}} = \frac{d\gamma^k(x)}{dx}$$

Levy's concentration lemma

Levy's concentration lemma

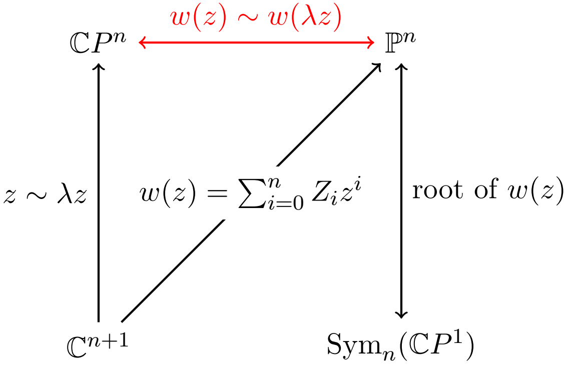
Let $f : S^{n-1} \to \mathbb{R}$ be a η-Lipschitz function. Let M_f denote the median of f and $\langle f \rangle$ denote the mean of f. (Note this can be generalized to many other manifolds (spaces that locally resembles Euclidean space).) Select a random point $x \in S^{n-1}$ with $n > 2$ according to the uniform measure (Haar measure). Then the probability of observing a value of f much different from the reference value is exponentially small.

$$\Pr[|f(x) - M_f| > \epsilon] \leq \exp(-\frac{n\epsilon^2}{2\eta^2})$$

$$\Pr[|f(x) - \langle f \rangle| > \epsilon] \leq 2\exp(-\frac{(n-1)\epsilon^2}{2\eta^2})$$

The Maxwell-Boltzmann distribution law will help us find the limit of measures on hemisphere S^{n-1} under the series of functions $f_n : S^{n-1}(\sqrt{n}) \to \mathbb{R}$.

Majorana stellar representation of the quantum state



Basically, there is a bijection between the complex projective space $\mathbb{C}P^n$ and the set of roots of a polynomial of degree n.
We can use a symmetric group of permutation of n complex numbers (or S^2) to represent the $\mathbb{C}P^n$, that is $\mathbb{C}P^n = S^2 \times S^2 \times \cdots \times S^2 / S_n$.

Future Plans

- The physical meaning of the mathematical structures, the correspondence, and the relationship between the measures, quantum states, and the geometry of topological spaces.
  - Fiber bundles
  - Fubini-Study metric
  - Space of entangled states
- Riemannian geometry of $\mathbb{C}P^n$.
  - Ricci curvature
  - Levy's Isoperimetric inequality
  - Lipschitz constants and Levi-Civita connection
  - Local operations and classical communication (LOCC)
- The proof of the Page's formula.
- Majorana stellar representation of the quantum state. And possibly the concentration of measure effect on that.
- Relations to Gromov's works [Gromov, 1981]
  - Levy families
  - Observable diameters

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Q&A